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Almost-Sure Stochastic Stability of Viscoelastic Plates in Supersonic Flow

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The dynamic stability of a viscoelastic plate in a supersonic gas flow and subjected to a stochastically fluctuating axial thrust is performed within the concept of the Lyapunov exponent. The constitutive relation is modeled in an integral form by using the Boltzmann superposition principle. The piston theory as a quasi-first-order approximation is used to represent the aerodynamic loading on the plate. The stochastic averaging method is used and the Khasminskii's technique [Khasminskii, R. A., "Necessary and Sufficient Conditions for the Asymptotic Stability of Linear Stochastic System," ***Theory of Probability and Its Application,*** **Vol. 12, No. 1, 1967, pp. 144–147 (English translation)] is employed to obtain the stability boundaries. The influence of the various plate and flow parameters and the random loading spectral densities on the stability are investigated.**

Introduction

THE dynamic stability of plates in a supersonic gas flow and subjected to in-plane loading is one of the most interesting problems in the field of structural vibrations. This phenomenon is observed, for example, in the instability of aircraft in airflow (wing flutter). Panel flutter is a form of dynamical instability resulting from the dynamic interaction of aerodynamic, inertia, and elastic forces and is defined as the self-excited oscillation of the external surface of the structural system. One of the difficulties in studying this phenomenon arises because aerodynamic forces cannot, in general, be simply expressed in terms of disturbances of the surface exposed to the flow.

Since the existence of panel flutter was established, various approximate expressions for the aerodynamic loading have been used. The application of a two-dimensional static approximation was used by Hedgepeth,[1] and because it was valid only for a small range of Mach numbers and panel geometry, a new detailed solution of the panel flutter problem was initiated by including three-dimensional unsteady aerodynamics. Expressions for the latter approximation are complicated and, thus, are of limited application. The simplest variant of the aerodynamic approximation is known as the law of plane sections or the piston theory, which provides a formula relating the aerodynamic pressure on the structure to the normal component of the velocity at any point considered. Using this approximation, Bolotin and Zhinzher,[2] Volmir,[3] and Dowell[4] extensively investigated the stability of elastic plates under periodic loading when the behavior is governed by the Mathieu equation, the stability boundaries being characterized by the Strutt diagram. When the plate material is of a linear viscoelastic type, the equation of motion becomes more complicated and turns out to be an integro–differential equation rather than an ordinary differential equation as in the elastic case. The stability of viscoelastic rods under deterministic loading was investigated by Matyash[5] by using the averaging method and by Stevens,[6] Szyskowski and Glockner,[7] and Glockner and Szyskowski[8] using a spring–dashpot representation for the viscoelastic constitutive relation.

Stability of long elastic plates in supersonic gas flow and subjected to in-plane ergodic stationary Gaussian stochastic loading was investigated in the work of Plaut and Infante,[9] Kozin,[10] and Ahmadi,[11] who obtained sufficient conditions for almost-sure stability of plates on the basis of Lyapunov's second method. Potapov[12] and Potapov and Bonder[13] treated the viscoelastic case and also obtained stability conditions in the first and second moments. Using a two-mode approximation, the stochastic nonlinear flutter of elastic plates was examined by Ibrahim et al.[14] and Ibrahim and Orono.[15] The in-plane excitation was assumed as a Gaussian white noise process, and the response moment equations were generated by making use of the Fokker–Planck equation approach. A cumulant closure approximation was then employed to truncate the moment equations.

When the excitation is nonwhite, the solution process is not a Markov diffusion process. Stratonovich[16] and Khasminskii[17] showed that, when the excitation has a small correlation time as compared to the relaxation time of the system, a physical non-Markov diffusion process can be approximated in the weak sense by a Markov diffusion process whose governing Itô equations are obtained by making use of the so-called stochastic averaging method, provided the limits of the averaged physical system equations exist. In this paper, the stochastic averaging method and the method of Larianov[18] for averaging the integral term that arises from the viscoelastic effect together with a technique first proposed by Khasminskii[19] are employed to obtain explicit expressions for the largest Lyapunov exponent that indicates the stability condition of the viscoelastic plate. It has been shown by Arnold and Kliemann[20] that the Lyapunov exponents are analogous to the real part of the eigenvalues of deterministic time-invariant systems. If the maximum exponent is positive, the system is unstable with probability one, and if it is negative, the system is stable with probability one. Hence, the vanishing of the maximum Lyapunov exponent indicates the transition state and, thus, the stability boundaries in parameter space. Numerical results are presented to give a quantitative picture of the effect of the plate and the nondimensional flow parameters, as well as the load spectral densities, on the stability boundaries.

Formulation

Consider a long viscoelastic plate, one side of which is exposed to a supersonic flow of gas, performing a small oscillation perpendicular to its plane. The plate is assumed to be effectively infinitely long and freely supported along the long edges (Fig. 1). A uniform thrust $N(t)$ per unit length is applied to the mobile edge at the midplane. Let $w(x,t)$ be the transverse deflection, where x is the longitudinal distance from one edge. By making use of piston theory for the approximation of the aerodynamic loading, the equation of motion of an elastic plate with small oscillation can be written as[21]

$$\rho h \frac{\partial^2 w(x,t)}{\partial t^2} + \rho h c \frac{\partial w(x,t)}{\partial t} + N(t)\frac{\partial^2 w(x,t)}{\partial x^2} - \frac{\partial^2 M_x}{\partial x^2} + P(x,t) = 0 \tag{1}$$

where ρ, h, and c are the plate material density, the plate thickness, and the viscous damping coefficient per unit mass, respectively, and $P(x,t)$ is an approximation to the component of the aerodynamic

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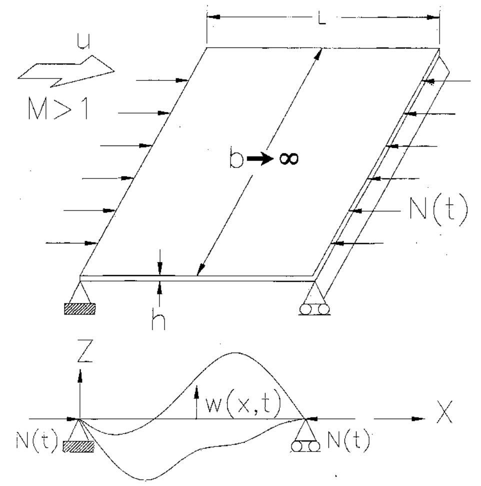

Fig. 1 Model of the plate.

force per unit area caused by the deviation of the plate from its undisturbed state,

$$P(x,t) = \chi \frac{P_\infty}{C_\infty}\left[\frac{\partial w(x,t)}{\partial t} + u\frac{\partial w(x,t)}{\partial x}\right] \tag{2}$$

where χ, P_∞, C_∞, and u represent gas flow polytropy index, undisturbed gas pressure, undisturbed gas speed, and flow velocity, respectively. With the use of Boltzmann's superposition principle, the constitutive relation for a linear, isotropic, viscoelastic material can be given by[22]

$$\sigma = 3\kappa\epsilon, \qquad \hat{s} = 2G(1 - \boldsymbol{R}^*)\hat{e} \tag{3}$$

where σ and ϵ and $\hat{s}$ and $\hat{e}$ are the first invariants and the deviatoric parts of the stress and strain tensors, respectively, and $\boldsymbol{R}^*$ is a relaxation operator given by

$$\boldsymbol{R}^*\psi = \int_0^t R^*(t-\tau)\psi(\tau)\,\mathrm{d}\tau, \qquad 0 \le \int_0^\infty R^*(\vartheta)\,\mathrm{d}\vartheta < 1$$

and κ and G denote the elastic bulk and shear moduli, respectively. In the case of cylindrical bending in the x direction, ϵ_x is the only nonzero component of the strain tensor, and $\epsilon_y = \epsilon_z = 0$. Therefore, the first strain invariant ϵ and the deviatoric part $\hat{e}$ are given by

$$\epsilon = \tfrac{1}{3}\epsilon_x, \qquad \hat{e}_x = \tfrac{2}{3}\epsilon_x, \qquad \hat{e}_y = \hat{e}_z = -(\epsilon_x/3) \tag{4}$$

Using the relation

$$\hat{\sigma} = \tfrac{1}{3}\sigma I + \hat{s} \tag{5}$$

where I is the unit tensor, one obtains

$$\sigma_x = \frac{1}{3}(\kappa + 4G)\left[\epsilon_x - \int_0^t \frac{4G}{(\kappa+4G)}R^*(t-\tau)\epsilon_x\,\mathrm{d}\tau\right] \tag{6}$$

Using Kirchchoff's hypothesis for the bending of plates

$$\epsilon_x = -z\frac{\partial^2 w}{\partial x^2} \tag{7}$$

and the bending moment per unit length M_x is given by

$$M_x = \int_{-h/2}^{h/2} z\sigma_x\,\mathrm{d}A = -D\left[\frac{\partial^2 w}{\partial x^2} - \int_0^t \tilde{R}(t-\tau)\frac{\partial^2 w}{\partial x^2}\,\mathrm{d}\tau\right] \tag{8}$$

where with the relations

$$\kappa = E/[3(1-2\nu)], \qquad G = E/[2(1+\nu)] \tag{9}$$

with E and ν as the initial constant Young's modulus and Poisson's ratio, respectively, D and $\tilde{\boldsymbol{R}}$ can be given by the following expressions:

$$D = \frac{(7-11\nu)Eh^3}{108(1-2\nu)(1+\nu)}, \qquad \tilde{\boldsymbol{R}} = \frac{6(1-2\nu)}{(7-11\nu)}\boldsymbol{R}^* \tag{10}$$

When M_x is substituted into Eq. (1), the equation of motion for a viscoelastic plate is given by

$$D(1-\tilde{\boldsymbol{R}})\frac{\partial^4 w(x,t)}{\partial x^4} + N(t)\frac{\partial^2 w(x,t)}{\partial x^2} + \rho h\frac{\partial^2 w(x,t)}{\partial t^2} + \rho hc\frac{\partial w(x,t)}{\partial t} + P(x,t) = 0 \tag{11}$$

When K is taken as the summation of the plate structural and aerodynamic damping, that is,

$$K = \rho hc + \chi(P_\infty/C_\infty) \tag{12}$$

and the Galerkin approximation is used,

$$w(x,\,t) = \sum_{n=1}^{m} f_n(t)\sin\frac{n\pi x}{L} \tag{13}$$

which satisfies the boundary conditions of simple support,

$$w(0,\,t) = w(L,\,t) = \left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=0,L} = 0 \tag{14}$$

Eq. (11) becomes

$$\ddot{f}_n + \frac{K}{\rho h}\dot{f}_n + \frac{D}{\rho h}\left(\frac{n\pi}{L}\right)^4\left[(1-\tilde{\boldsymbol{R}}) - \frac{\xi^*(t)}{n^2}\right]f_n + \frac{1}{\rho h}\sum_{j=1}^{m} b^*_{nj}f_j = 0, \qquad n = 1,2,\ldots,m \tag{15}$$

where

$$\xi^*(t) = \frac{L^2 N(t)}{\pi^2 D}$$

$$b^*_{nj} = \frac{4\chi P_\infty u}{C_\infty L}\begin{cases} nj/(n^2-j^2) & \text{if } (n\pm j) \text{ is odd} \\ 0 & \text{if } (n\pm j) \text{ is even}\end{cases} \tag{16}$$

It has been shown by Bolotin and Petrovsky[23] that not only qualitative, but, to some extent, quantitative results are predicted rather reliably with the use of the first two modes. Hence, considering $m = 2$, the reduced equations with $K = 0$, $\tilde{\boldsymbol{R}} = \boldsymbol{0}$, and $\xi^* = 0$ are

$$\ddot{f}_1 + \frac{D}{\rho h}\left(\frac{\pi}{L}\right)^4 f_1 - \frac{8\chi P_\infty u}{3C_\infty \rho hL}f_2 = 0$$

$$\ddot{f}_2 + \frac{D}{\rho h}\left(\frac{\pi}{L}\right)^4 f_2 + \frac{8\chi P_\infty u}{3C_\infty \rho hL}f_1 = 0 \tag{17}$$

When the nondimensional time t_1 is introduced by

$$t_1 = (\pi^2/L^2)\sqrt{(D/\rho h)}t \tag{18}$$

and the prime is used to denote differentiation with respect to the new time t_1, Eq. (17) become

$$\boldsymbol{f}'' + A\boldsymbol{f} = \boldsymbol{0} \tag{19}$$

where

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \qquad A = \begin{bmatrix} 1 & -\alpha \\ \alpha & 16 \end{bmatrix}, \qquad \alpha = \frac{8\chi P_\infty uL^3}{3C_\infty \pi^4 D} \tag{20}$$

For $\alpha \le \alpha_{\mathrm{cr}} = \frac{15}{2}$, the vibration frequencies obtained from Eq. (19) are real and are given by

$$\omega^2_{1,2} = \tfrac{17}{2} \mp \left[\left(\tfrac{15}{2}\right)^2 - \alpha^2\right]^{\frac{1}{2}} \tag{21}$$

When the transformation $\boldsymbol{f} = T\boldsymbol{q}$ is employed, where

$$T = \begin{bmatrix} -\alpha & -\alpha c^* \\ (\omega_1^2 - 1) & c^*(\omega_2^2 - 1) \end{bmatrix} \tag{22}$$

and where c^* is a parameter to be chosen to obtain a suitable coordinate scaling, Eqs. (15) with $m = 2$ transform to

$$T\boldsymbol{q}'' + AT\boldsymbol{q} = -2\beta T\boldsymbol{q}' + n^4\boldsymbol{R}(T\boldsymbol{q}) + BT\boldsymbol{q} \tag{23}$$

where

$$\xi(t_1) = \xi^*(t), \qquad \boldsymbol{R}(t_1) = \tilde{\boldsymbol{R}}(t)$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\xi(t_1), \qquad \beta = \frac{KL^2}{2\pi^2\sqrt{\rho h D}}$$

By substitution for matrices A, T, and B, Eq. (23) becomes

$$q_1'' + 2\beta q_1' + \omega_1^2 q_1 - \int_0^{t_1} R(t_1 - \tau) q_1(\tau)\,\mathrm{d}\tau + \omega_1(k_{11}q_1 + k_{12}q_2)\xi(t_1) = 0$$

$$q_2'' + 2\beta q_2' + \omega_2^2 q_2 - 16\int_0^{t_1} R(t_1 - \tau) q_2(\tau)\,\mathrm{d}\tau + \omega_2(k_{21}q_1 + k_{22}q_2)\xi(t_1) = 0 \tag{24}$$

where

$$k_{11} = \frac{(4\omega_1^2 - \omega_2^2 - 3)}{\omega_1(\omega_2^2 - \omega_1^2)}, \qquad k_{12} = \frac{3c^*(\omega_2^2 - 1)}{\omega_1(\omega_2^2 - \omega_1^2)}$$

$$k_{21} = \frac{-3(\omega_1^2 - 1)}{c^*\omega_2(\omega_2^2 - \omega_1^2)}, \qquad k_{22} = \frac{(\omega_1^2 - 4\omega_2^2 + 3)}{\omega_2(\omega_2^2 - \omega_1^2)}$$

For convenience, the parameter c^* is chosen such that $k_{12} = -k_{21}$, which gives

$$c^* = \frac{(15 - \alpha_0)^{\frac{1}{2}}(17 - \alpha_0)^{\frac{1}{4}}}{(15 + \alpha_0)^{\frac{1}{2}}(17 + \alpha_0)^{\frac{1}{4}}}$$

where

$$\alpha_0 = (\omega_2^2 - \omega_1^2) = 2(\alpha_{\mathrm{cr}}^2 - \alpha^2)^{\frac{1}{2}} \tag{25}$$

Substituting for $\omega_{1,2}$ and c^*, one obtains

$$k_{11} = \frac{5(9 - \alpha_0)}{\sqrt{2}\alpha_0(17 - \alpha_0)^{\frac{1}{2}}}, \qquad k_{12} = \frac{3(15^2 - \alpha_0^2)^{\frac{1}{2}}}{\sqrt{2}\alpha_0(17^2 - \alpha_0^2)^{\frac{1}{4}}}$$

$$k_{21} = \frac{-3(15^2 - \alpha_0^2)^{\frac{1}{2}}}{\sqrt{2}\alpha_0(17^2 - \alpha_0^2)^{\frac{1}{4}}}, \qquad k_{22} = \frac{-5(9 + \alpha_0)}{\sqrt{2}\alpha_0(17 + \alpha_0)^{\frac{1}{2}}}$$

Henceforth, we shall denote $k_{12} = -k_{21}$ by k.

Approximation to Markov Process

The system of Eqs. (24) obviously admits the trivial solution $q_1 = q_2 = 0$. If $\xi(t_1)$ is taken to be an ergodic stochastic process with zero mean value and sufficiently small correlation time, and also if the damping term β, the cosine and sine spectral densities of $\xi(t_1)$, respectively, $S(\omega)$ and $\Psi(\omega)$, are small and of the same order, that is, $\beta = \mathcal{O}(\varepsilon)$ and $S(\omega) = \mathcal{O}(\varepsilon)$ and $\Psi(\omega) = \mathcal{O}(\varepsilon)$ and $0 < \varepsilon \ll 1$, then the stochastic averaging method may be used to replace Eqs. (24) by approximate Itô equations. The cosine and sine spectral densities are defined as

$$S(\omega) = 2\int_0^\infty E[\xi(t_1)\xi(t_1 + \tau)]\cos\omega\tau\,\mathrm{d}\tau$$

$$\Psi(\omega) = 2\int_0^\infty E[\xi(t_1)\xi(t_1 + \tau)]\sin\omega\tau\,\mathrm{d}\tau$$

where $E[\cdot]$ denotes the expectation operator. With the transformation

$$q_i = a_i\cos\Theta_i, \qquad q'_i = -a_i\omega_i\sin\Theta_i$$

$$\Theta_i = \omega_i t_1 + \theta_i, \qquad i = 1, 2 \tag{26}$$

and the method of variation of parameters, the following four first-order equations in a_1, θ_1, a_2, and θ_2 are obtained:

$$\dot{a}_1 = \frac{1}{2}k_{11}a_1\xi(t_1)\sin 2\Theta_1 + k_{12}a_2\xi(t_1)\sin\Theta_1\cos\Theta_2 - 2\beta a_1\sin^2\Theta_1 - \frac{\sin\Theta_1}{\omega_1}\int_0^{t_1} R(t_1 - \tau)a_1\cos\Theta_1(\tau)\,\mathrm{d}\tau$$

$$\dot{\theta}_1 = k_{11}\xi(t_1)\cos^2\Theta_1 + k_{12}\frac{a_2}{a_1}\xi(t_1)\cos\Theta_1\cos\Theta_2 - \beta\sin 2\Theta_1 - \frac{\cos\Theta_1}{\omega_1}\int_0^{t_1} R(t_1 - \tau)\cos\Theta_1(\tau)\,\mathrm{d}\tau$$

$$\dot{a}_2 = k_{21}a_1\xi(t_1)\cos\Theta_1\sin\Theta_2 + k_{22}\frac{a_2}{2}\xi(t_1)\sin 2\Theta_2 - 2\beta a_2\sin^2\Theta_2 - \frac{16\sin\Theta_2}{\omega_2}\int_0^{t_1} R(t_1 - \tau)a_2\cos\Theta_2(\tau)\,\mathrm{d}\tau$$

$$\dot{\theta}_2 = k_{21}\frac{a_1}{a_2}\xi(t_1)\cos\Theta_1\cos\Theta_2 + k_{22}\xi(t_1)\cos^2\Theta_2 - \beta\sin 2\Theta_2 - \frac{16\cos\Theta_2}{\omega_2}\int_0^{t_1} R(t_1 - \tau)\cos\Theta_2(\tau)\,\mathrm{d}\tau \tag{27}$$

To ensure that the frequency difference $(\omega_2 - \omega_1)$ is not small, an appropriate range of the nondimensional flow parameter α_0, below the critical value α_{cr}, is considered for this analysis. A sufficient frequency difference is required so that the averaged amplitude equations $a_i(t)$ can become uncoupled from those for the phase terms $\theta_i(t)$. As ε decreases, the solution of the system of Eqs. (27) converges in the weak sense and up to first order in ε to a diffusive Markov process, whose governing Itô equations are of the form

$$\mathrm{d}a_i = m_i\,\mathrm{d}t + \sum_{j=1}^{2}\sigma_{ij}\,\mathrm{d}W_{a_j}$$

$$\mathrm{d}\theta_i = n_i\,\mathrm{d}t + \sum_{j=1}^{2}\mu_{ij}\,\mathrm{d}W_{\theta_j}, \qquad i = 1, 2 \tag{28}$$

where W_{a_j} and W_{θ_j} are mutually independent unit Wiener processes. The most important feature of the stochastic averaging method in the present problem is that the limiting averaged amplitudes processes $a_i(t)$ are decoupled from those of the phase angle processes $\theta_i(t)$. By making use of this property, the investigation from now on will only consider the averaged amplitudes $a_i(t)$. If the relaxation function $R(t)$ is integrable such that

$$\int_0^\infty R(t)\,\mathrm{d}t < \infty$$

$$\int_0^\infty tR(t)\,\mathrm{d}t < \infty$$

and, furthermore, $R(t) = \mathcal{O}(\varepsilon)$, the method of Larianov[18] can be used to average the integral term involving the relaxation measure. With the use of the stochastic averaging method and Larianov's method, the following expressions are obtained:

$$m_1 = \left[-\beta - \frac{R_s(\omega_1)}{2\omega_1} + \frac{3}{16}k_{11}^2 S(2\omega_1) - \frac{k^2}{8}S^-\right]a_1 + \frac{k^2 S^+}{16}\frac{a_2^2}{a_1}$$

$$m_2 = \left[-\beta - \frac{16R_s(\omega_2)}{2\omega_2} + \frac{3}{16}k_{22}^2 S(2\omega_2) - \frac{k^2}{8}S^-\right]a_2 + \frac{k^2 S^+}{16}\frac{a_1^2}{a_2}$$

$$(\sigma\sigma^T)_{11} = \frac{k_{11}^2}{8}S(2\omega_1)a_1^2 + \frac{k^2}{8}S^+ a_2^2$$

$$(\sigma\sigma^T)_{22} = \frac{k_{22}^2}{8}S(2\omega_2)a_2^2 + \frac{k^2}{8}S^+ a_1^2$$

$$(\sigma\sigma^T)_{12} = (\sigma\sigma^T)_{21} = -\frac{k^2}{8}S^- a_1 a_2 \tag{29}$$

where $R_s(\omega_1)$, $R_s(\omega_2)$, and $S^\pm$ are defined as follows:

$$R_s(\omega_i) = \int_0^\infty R(s)\sin\omega_i s\,\mathrm{d}s$$

$$S^\pm = S(\omega_1 + \omega_2) \pm S(\omega_1 - \omega_2) \tag{30}$$

Lyapunov Exponent

The averaged amplitude vector (a_1, a_2) is a two-dimensional diffusion process, and the coefficients m_i and σ_{ij}, on the right-hand side of the first of Eqs. (28) are homogeneous in a_1 and a_2 of degree one. Therefore Khasmiskii's[19] technique may be employed to derive an expression for the largest Lyapunov exponent of the amplitude process. With the logarithmic polar transformation used,

$$\rho = \tfrac{1}{2}\log\left(a_1^2 + a_2^2\right), \qquad \phi = \tan^{-1}(a_2/a_1), \qquad 0 \le \phi \le \pi/2 \tag{31}$$

and Itô's differential rule used, the following pair of Itô equations governing ρ and ϕ are obtained:

$$\mathrm{d}\rho = Q(\phi)\,\mathrm{d}t + \Sigma(\phi)\,\mathrm{d}W, \qquad \mathrm{d}\phi = \Phi(\phi)\,\mathrm{d}t + \Psi(\phi)\,\mathrm{d}W \tag{32}$$

where the coefficient functions in Eqs. (32) are given by

$$Q(\phi) = \lambda_1\cos^2\phi + \lambda_2\sin^2\phi - (k^2/8)S^- + \Psi^2(\phi)$$

$$\Sigma^2(\phi) = \tfrac{1}{8}\left[k_{11}^2 S(2\omega_1)\cos^4\phi + k_{22}^2 S(2\omega_2)\sin^4\phi\right] + \tfrac{1}{8}k^2 S(\omega_1 - \omega_2)\sin^2 2\phi$$

$$\Phi(\phi) = \tfrac{1}{64}\left[k_{11}^2 S(2\omega_1)_1 + k_{22}^2 S(2\omega_2)\right]\sin 4\phi - \tfrac{1}{16}k^2 S(\omega_1 - \omega_2)\sin 4\phi + (k^2/8)S^+\cot 2\phi - \tfrac{1}{2}(\lambda_1 - \lambda_2)\sin 2\phi$$

$$\Psi^2(\phi) = \tfrac{1}{32}\left[k_{11}^2 S(2\omega_1) + k_{22}^2 S(2\omega_2)\right]\sin^2 2\phi - \tfrac{1}{8}k^2 S(\omega_1 - \omega_2)\sin^2 2\phi + (k^2/8)S^+ \tag{33}$$

The constants λ_1 and λ_2 are defined by

$$\lambda_1 = -\beta - [R_s(\omega_1)/2\omega_1] + \left(k_{11}^2/8\right)S(2\omega_1)$$

$$\lambda_2 = -\beta - [16R_s(\omega_2)/2\omega_2] + \left(k_{22}^2/8\right)S(2\omega_2)$$

which, as will be shown later, are the Lyapunov exponents of the uncoupled systems that result when $k_{12} = k_{21} = 0$. The second of Eqs. (32) shows that the ϕ process is itself a diffusion process on the first quadrant of the unit circle.

Nonsingular Case

When the diffusion coefficient $\Psi(\phi)$ of the diffusion phase process $\phi(t)$ does not vanish in $0 \le \phi \le \pi/2$, the process $\phi(t)$ is nonsingular and the density $\mu(\phi)$ of its invariant measure is governed by the following Fokker–Planck equation:

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\left\{\Phi(\phi)\mu(\phi) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\phi}[\Psi^2(\phi)\mu(\phi)]\right\} = 0 \tag{34}$$

The general solution to the Fokker–Plank equation (34) is

$$\mu(\phi) = \frac{C}{\Psi^2(\phi)U(\phi)} - \frac{G_0}{\Psi^2(\phi)U(\phi)}\int U(\phi)\,\mathrm{d}\phi \tag{35}$$

where C and G_0 are integration constants and

$$U(\phi) = \exp\left[-2\int\{\Phi(\phi)\Psi^{-2}(\phi)\}\,\mathrm{d}\phi\right] \tag{36}$$

Substituting for $\Phi(\phi)$ and $\Psi^2(\phi)$ from Eqs. (33) into Eq. (36) and letting

$$b = \tfrac{1}{32}\left[k_{11}^2 S(2\omega_1) + k_{22}^2 S(2\omega_2) - 4k^2 S(\omega_1 - \omega_2)\right]$$

$$a = \tfrac{1}{32}\left[k_{11}^2 S(2\omega_1) + k_{22}^2 S(2\omega_2) + 4k^2 S(\omega_1 + \omega_2)\right]$$

one obtains

$$U(\phi) = \frac{1}{\sin 2\phi}\exp\left[\frac{\lambda_2 - \lambda_1}{2a}\int^{\cos 2\phi}\frac{\mathrm{d}t}{1 - (b/a)t^2}\right] \tag{37}$$

Because a is always positive, the preceding integration depends on the sign of the constant b.

For no accumulation of probability mass at the boundaries, the stationary probability flux represented by G_0 has to be zero, and, therefore, the ϕ process is ergodic in the interval $0 \le \phi \le \pi/2$. The invariant density $\mu(\phi)$ is then given by

$$\mu(\phi) = C/[\Psi^2(\phi)U(\phi)] \tag{38}$$

where C is determined from the normalization condition

$$\int_0^{\pi/2}\mu(\phi)\,\mathrm{d}\phi = 1 \tag{39}$$

Performing the integration in Eq. (37), one obtains the following. For $b > 0$,

$$\mu(\phi) = \frac{C\sin 2\phi}{\Psi^2(\phi)}\exp\left[\frac{\lambda_1 - \lambda_2}{2\sqrt{\Delta}}\tanh^{-1}\frac{b\cos 2\phi}{\sqrt{\Delta}}\right] \tag{40}$$

For $b < 0$,

$$\mu(\phi) = \frac{C\sin 2\phi}{\Psi^2(\phi)}\exp\left[-\frac{\lambda_1 - \lambda_2}{2\sqrt{-\Delta}}\tan^{-1}\frac{b\cos 2\phi}{\sqrt{-\Delta}}\right] \tag{41}$$

For $b = 0$,

$$\mu(\phi) = \frac{C\sin 2\phi}{\Psi^2(\phi)}\exp\left[\frac{(\lambda_1 - \lambda_2)\cos 2\phi}{2a}\right] \tag{42}$$

where $\Delta = ab$ and the normalization constant C is given by the following. For $b > 0$,

$$C = \frac{1}{2}(\lambda_1 - \lambda_2)\mathrm{csch}\left(\frac{\lambda_1 - \lambda_2}{2\sqrt{\Delta}}\tanh^{-1}\frac{b}{\sqrt{\Delta}}\right) \tag{43}$$

For $b < 0$,

$$C = \frac{1}{2}(\lambda_1 - \lambda_2)\mathrm{csch}\left(\frac{\lambda_1 - \lambda_2}{2\sqrt{-\Delta}}\tan^{-1}\frac{b}{\sqrt{-\Delta}}\right) \tag{44}$$

For $b = 0$,

$$C = \frac{1}{2}(\lambda_1 - \lambda_2)\mathrm{csch}\left(\frac{\lambda_1 - \lambda_2}{2a}\right) \tag{45}$$

With Khasmiskii's[19] procedure used, the largest Lyapunov exponent of the amplitude process is given by

$$\lambda = E[Q(\phi)] = \int_0^{\pi/2} Q(\phi)\mu(\phi)\,\mathrm{d}\phi \tag{46}$$

with probability one (WP1). Substituting for $Q(\phi)$ and $\mu(\phi)$ and performing the indicated integration yield the following expressions for the largest Lyapunov exponent. For $b > 0$,

$$\lambda = \frac{1}{2}(\lambda_1 - \lambda_2)\coth\left(\frac{\lambda_1 - \lambda_2}{2\sqrt{\Delta}}\tanh^{-1}\frac{b}{\sqrt{\Delta}}\right) + \frac{1}{2}(\lambda_1 + \lambda_2) - \frac{k^2}{8}S^- \tag{47}$$

For $b < 0$,

$$\lambda = \frac{1}{2}(\lambda_1 - \lambda_2)\coth\left(\frac{\lambda_1 - \lambda_2}{2\sqrt{-\Delta}}\tan^{-1}\frac{-b}{\sqrt{-\Delta}}\right) + \frac{1}{2}(\lambda_1 + \lambda_2) - \frac{k^2}{8}S^- \tag{48}$$

For $b = 0$,

$$\lambda = \frac{1}{2}(\lambda_1 - \lambda_2)\coth\left(\frac{4(\lambda_1 - \lambda_2)}{k^2 S^+}\right) + \frac{1}{2}(\lambda_1 + \lambda_2) - \frac{k^2}{8}S^- \tag{49}$$

The expressions for the largest Lyapunov exponent can be simplified to another, more convenient, form.

Let $\eta_0 = \frac{1}{2}\tanh^{-1}(b/\sqrt{\Delta})$ so that $\cosh\eta_0 = (a+b)/(a-b)$. Substituting for a and b and taking $\Delta_0 = 16\Delta$ give

$$\cosh\eta_0 = \frac{\left[k_{11}^2 S(2\omega_1) + k_{22}^2 S(2\omega_2) + 2k^2 S^-\right]}{2k^2 S^+} \tag{50}$$

Therefore, for $b > 0$,

$$\lambda = \frac{1}{2}(\lambda_1 - \lambda_2)\coth\left(\frac{\lambda_1 - \lambda_2}{\sqrt{\Delta_0}}\eta_0\right) + \frac{1}{2}(\lambda_1 + \lambda_2) - \frac{k^2}{8}S^- \tag{51}$$

For $b < 0$, let

$$\eta_0 = \tfrac{1}{2}\tan^{-1}(b/\sqrt{-\Delta})$$

so that

$$\eta_0 = \cos^{-1}\frac{\left[k_{11}^2 S(2\omega_1) + k_{22}^2 S(2\omega_2) + 2k^2 S^-\right]}{2k^2 S^+} \tag{52}$$

and then

$$\lambda = \frac{1}{2}(\lambda_1 - \lambda_2)\coth\left(\frac{\lambda_1 - \lambda_2}{\sqrt{-\Delta_0}}\eta_0\right) + \frac{1}{2}(\lambda_1 + \lambda_2) - \frac{k^2}{8}S^- \tag{53}$$

When β, ω_1, ω_2, k_{11}, k_{22}, k, Δ, and η_0, are substituted for, where η_0 is given by

$$\eta_0 = \begin{cases} \cosh^{-1}[(a+b)/(a-b)] & \text{for } b > 0 \\ \cos^{-1}[(a+b)/(a-b)] & \text{for } b < 0 \end{cases} \tag{54}$$

stability conditions in terms of the various plate and flow parameters, as well as the excitation spectral densities, can be obtained.

Singular Case

When the diffusion coefficient $\Psi(\phi)$ of the diffusion phase process $\phi(t)$ vanishes at some point $\phi(t) = \phi_0$ in the interval $[0, \pi/2]$, the diffusion process is not ergodic in the whole interval and is considered to be singular. From the last of Eqs. (33), for $\Psi^2(\phi)$ to vanish at $\phi(t) = \phi_0 = \pi/4$, one of the following sets of conditions must be satisfied:

$$k_{11} = 0, \qquad S(2\omega_2) = 0, \qquad S(\omega_1 + \omega_2) = 0$$

$$k_{22} = 0, \qquad S(2\omega_1) = 0, \qquad S(\omega_1 + \omega_2) = 0 \tag{55}$$

and for $\Psi^2(\phi)$ to vanish at $\phi(t) = \phi_0 = 0$ and $\pi/2$, the following conditions must be satisfied:

$$S(\omega_1 + \omega_2) = 0, \qquad S(\omega_1 - \omega_2) = 0 \tag{56}$$

The sign of the drift coefficient of the diffusion process has to be checked to determine the nature of the singular point (see, for example, Mitchell and Kozin[24]). On substituting into Eqs. (33) and by making use of one of the conditions of Eqs. (55) for a singular point at $\phi_0 = \pi/4$, we have three cases:

1) If $R_s(\omega_1)/2\omega_1 > 8R_s(\omega_2)/\omega_2$, then $\Phi(\pi/4) > 0$ and the singular point $\phi_0 = \pi/4$ is a right shunt or forward shunt. This means that even if an initial point $\phi(t)$ is in the left half interval $[0, \pi/4]$, it will eventually be shunted across to the right half interval $[\pi/4, \pi/2]$ and remain there forever. The density $\mu(\phi)$ is concentrated in the right half of the interval $0 \le \phi \le \pi/2$ and is governed by the Fokker–Planck equation (34) whose solution now is of the form

$$\mu(\phi) = \begin{cases} 0 & 0 \le \phi < \pi/4 \\ C/[\Psi^2(\phi)U(\phi)] & \pi/4 \le \phi \le \pi/2 \end{cases} \tag{57}$$

When one of the conditions of Eqs. (55) is substituted into Eqs. (33) and Eqs. (37–39) and (46) are used, the following expression for the largest Lyapunov exponent can be obtained:

$$\lambda = -\beta - [8R_s(\omega_2)/\omega_2] + (k^2/8)S(\omega_1 - \omega_2) \tag{58}$$

2) If $R_s(\omega_1)/2\omega_1 < 8R_s(\omega_2)/\omega_2$, the drift coefficient $\Phi(\pi/4) < 0$ and, therefore, the singular point $\phi_0 = \pi/4$ is a left shunt or backward shunt. This means that even if a $\phi(t)$ is in the right-half interval $[\pi/4, \pi/2]$, it will eventually be shunted across to the left-half interval $[0, \pi/4]$ and remain there forever. The density $\mu(\phi)$ is concentrated in the left half of the interval $0 \le \phi \le \pi/2$ and is governed by the Fokker–Planck equation (34), whose solution is given by

$$\mu(\phi) = \begin{cases} C/[\Psi^2(\phi)U(\phi)] & 0 \le \phi \le \pi/4 \\ 0 & \pi/4 < \phi \le \pi/2 \end{cases} \tag{59}$$

Similarly, we can obtain the expression for the largest Lyapunov exponent as follows:

$$\lambda = -\beta - [R_s(\omega_1)/2\omega_1] + (k^2/8)S(\omega_1 - \omega_2) \tag{60}$$

3) If $R_s(\omega_1)/2\omega_1 = 8R_s(\omega_2)/\omega_2$, then $\Phi(\pi/4) = 0$ and the singular point $\phi_0 = \pi/4$ is a trap point. This implies that regardless of where the initial point $\phi(t)$ is situated, it will eventually be attracted to the point $\phi_0 = \pi/4$ and remain there forever. The density of the invariant measure $\mu(\phi)$ is the Dirac delta function concentrated at $\pi/4$ and is given by

$$\mu(\phi) = \delta[\phi - (\pi/4)], \qquad 0 \le \phi \le \pi/2 \tag{61}$$

With Eqs. (55) and (46) used, the largest Lyapunov exponent for the trap point case can be evaluated as

$$\lambda = -\beta - [R_s(\omega_1)/2\omega_1] + (k^2/8)S(\omega_1 - \omega_2) \tag{62}$$

Similarly, it can be shown that, for the uncoupled system that results when $k_{12} = k_{21} = 0$, the singular points at $\phi_0 = 0$ and $\phi_0 = \pi/2$ are trap points, and the corresponding densities $\mu(\phi)$ of the invariant measures are the Dirac delta functions $\delta(\phi)$ and $\delta(\phi - \pi/2)$. With Eq. (46) used, it can be shown that the largest Lyapunov exponents for the uncoupled cases corresponding to $\phi_0 = 0$ and $\phi_0 = \pi/2$ are equal to λ_1 and λ_2, respectively.

Stability Analysis

The trivial solution of Eqs. (24) is asymptotically stable WP1 if λ is negative and unstable if λ is positive. The region of almost-sure stability for the system of Eqs. (24) is determined by the condition $\lambda < 0$. The stability boundaries may be defined by setting $\lambda = 0$, which gives a relation among β, ω_1, ω_2, $R_s(\omega_1)$, $R_s(\omega_2)$, k_{11}, k_{22}, k, and the spectral density of the excitation $S(\omega_0)$, evaluated at $\omega_0 = 2\omega_1$, $2\omega_2$, and $\omega_1 \pm \omega_2$, because only the first-order approximation is considered. Here β, ω_1, ω_2, k_{11}, k_{22}, and k are functions of plate dimensions, plate material type, undisturbed gas condition, and flow velocity, whereas $R_s(\omega_1)$ and $R_s(\omega_2)$ are the one-sided Fourier sine transforms of the relaxation function $R(t_1)$ evaluated at the natural frequencies ω_1 and ω_2, respectively.

For the present analysis, a band-limited excitation and a white noise excitation are considered. It may be seen that only those values of the excitation spectrum at the frequencies $2\omega_1$, $2\omega_2$, and $\omega_1 \pm \omega_2$ have an effect on the stability condition. To show this more explicitly, some particular forms of the excitation spectrum are considered. For band-limited excitation, the spectral density is considered to be small everywhere when compared with those near the neighborhood of some frequency ω_0; thus, $S(\omega)$ is assumed to be concentrated in a narrow bandwidth, $\omega_0 - \Delta\omega_0/2 < \omega < \omega_0 + \Delta\omega_0/2$, where $\Delta\omega_0 \ll \omega_0$. For such a process with spectral density $S(\omega) = \mathcal{O}(\epsilon)$, $0 < |\epsilon| \ll 1$, the correlation time τ_c is $\mathcal{O}(1/\Delta\omega_0)$, whereas the relaxation time τ_r of the system of Eqs. (24) is $\mathcal{O}(1/\epsilon)$. Hence, if $\Delta\omega_0 \gg \epsilon$, then $\tau_c \ll \tau_r$, and the Markov process approximation made in the preceding sections will remain valid. In this analysis, the cases in which ω_0 lies in the neighborhood of $2\omega_1$, $2\omega_2$, and $\omega_1 \pm \omega_2$ are considered. For white noise excitation, the spectrum has a constant value for all frequencies ω. The band-limited excitation case gives a quantitative picture of the effect of the excitation spectrum on the almost-sure stability.

As an example, for obtaining some numerical results, polyurethane is considered as the plate material and, to justify using thin plate theory, a length–thickness ratio of $L/h = 100$ is considered. To ensure supersonic flow and an appropriate frequency difference, the nondimensional parameter α_0 is considered in the range $3 < \alpha_0 < 10.5$. The relaxation function is taken as

$$R(t) = \sum_{i=1}^{m} \zeta_i \chi_i^* e^{-\zeta_i t} \tag{63}$$

where χ_i^* and ζ_i^{-1} are the material characteristic viscosity and the characteristic time of relaxation, respectively. With $m = 1$, $\zeta_1 = 18.52$ s^{-1}, and $\chi^*_1 = 0.283$ (Ref. 25), the one-sided Fourier sine transform of the relaxation function can be derived as

$$R_S(\omega_i) = \frac{5.241\omega_i}{18.52^2 + \omega_i^2} \tag{64}$$

Stability boundaries for various spectral densities of a band-limited and a white noise excitation, calculated from the obtained expressions for the largest Lyapunov exponent, are presented in Figs. 2–7. In Fig. 2 it is shown that for a band-limited excitation, $S(2\omega_2)$

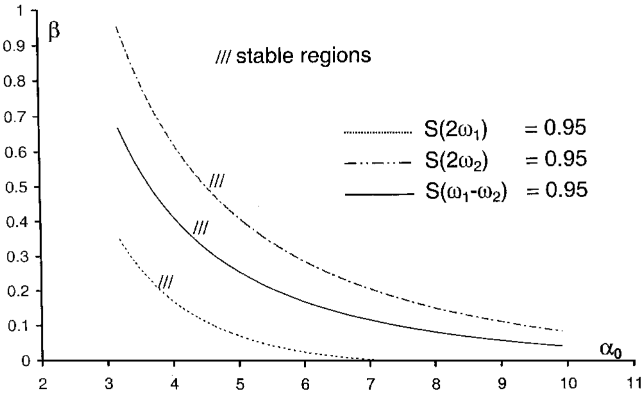

Fig. 2 Stability boundaries under a band-limited excitation.

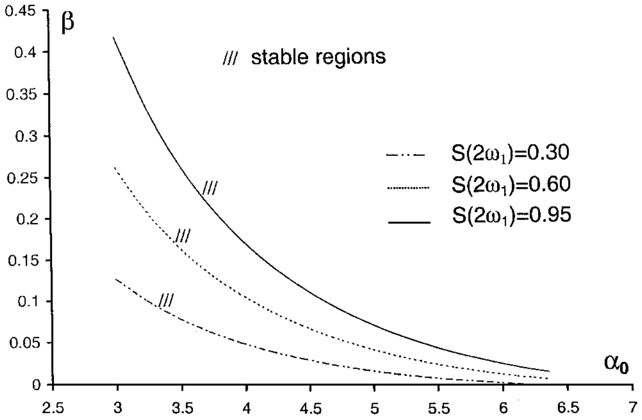

Fig. 3 Effect of $S(2\omega_1)$ on stability boundaries under a band-limited excitation.

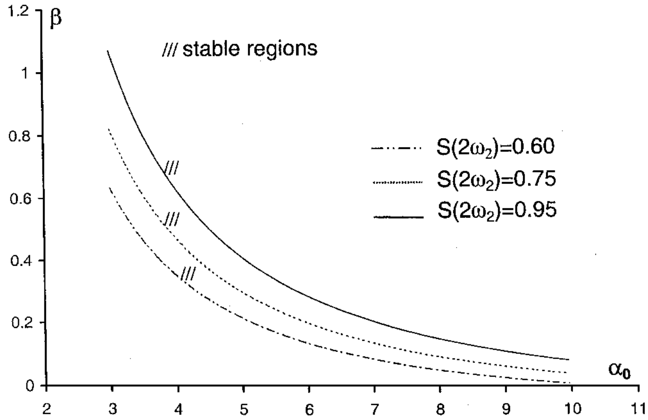

Fig. 4 Effect of $S(2\omega_2)$ on stability boundaries under a band-limited excitation.

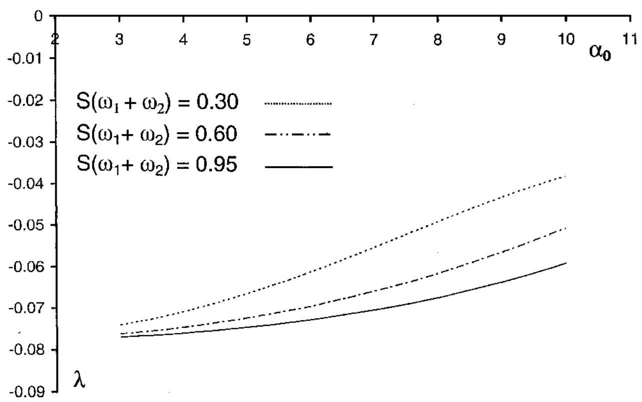

Fig. 5 Effect of $S(\omega_2 + \omega_1)$ on stability boundaries under a band-limited excitation.

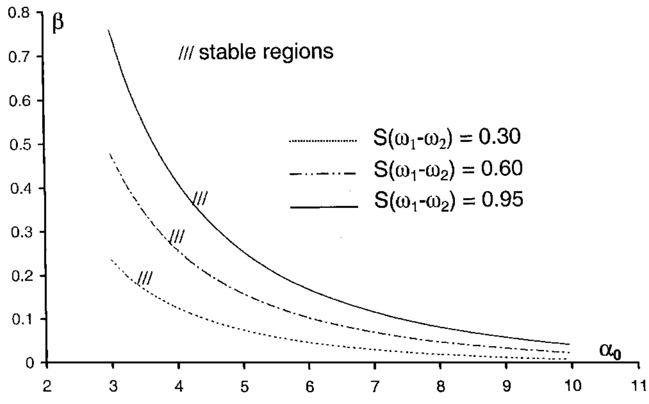

Fig. 6 Effect of $S(\omega_2 - \omega_1)$ on stability boundaries under a band-limited excitation.

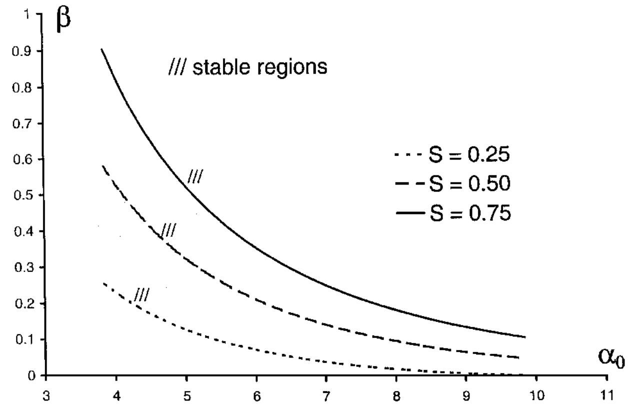

Fig. 7 Effect of spectral density S on stability boundaries under a white noise excitation.

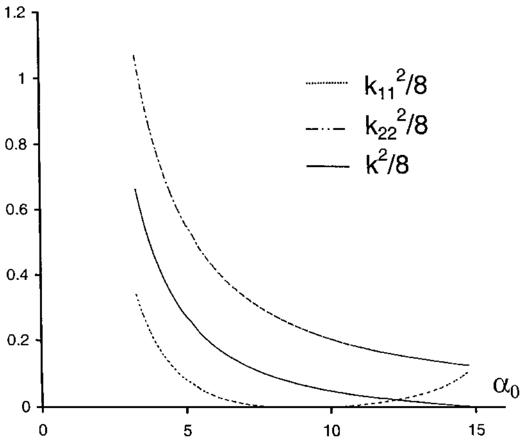

Fig. 8 Effect of the nondimensional flow parameter α_0 on the stiffness terms k_{11}, k_{22}, and k.

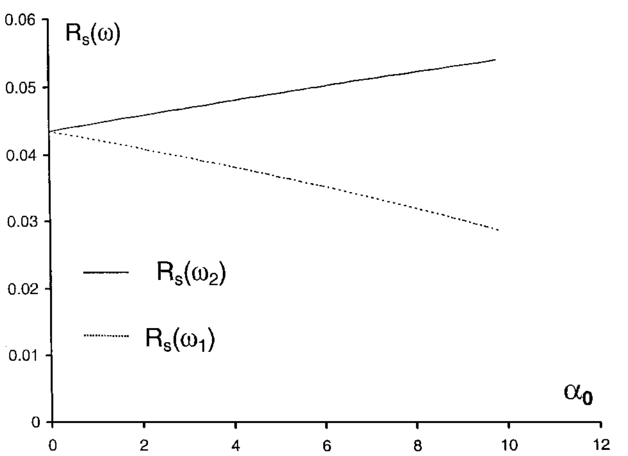

Fig. 9 Effect of the nondimensional flow parameter α_0 on the one-sided Fourier sine transform of the relaxation functions $R_s(\omega_1)$ and $R_s(\omega_2)$.

has the largest destabilizing effect because it is associated with the term k_{22}^2, which is dominant over the other two terms k_{11}^2 and k^2, whereas $S(2\omega_1)$ has the least destabilizing effect. Also note that the destabilizing effect for the different excitation values decreases with the increase of the nondimensional parameter α_0. From Figs. 3–6, it can be inferred that the variation of the spectral densities for the band-limited excitation at the frequencies $\omega_0 = 2\omega_1$, $2\omega_2$, and $\omega_1 - \omega_2$ have a significant effect on the stability boundaries and that $S(2\omega_1)$, $S(\omega_2)$, and $S(\omega_1 - \omega_2)$ have a destabilizing effect, whereas $S(\omega_1 + \omega_2)$ always has a stabilizing effect. It can also be inferred that more damping is required for stabilizing the system as the flow becomes more supersonic. From Fig. 7, it can be deduced that for the white noise excitation the spectral density S has a destabilizing effect and also that greater damping is needed for higher flow conditions. Figures 8 and 9 show the effect of the nondimensional flow parameter α_0 on the stiffness terms k_{11}, k_{22}, and k and on the one-sided Fourier sine transforms of the relaxation function $R_s(\omega_1)$ and $R_s(\omega_2)$, respectively.

Conclusions

Piston theory is used to give a quasi-steady first-order approximation for the aerodynamic loading on the plate. The viscoelastic constitutive relations of the plate material are represented in integral form by making use of the Boltzmann superposition principle. By the use of the Galerkin method, the equation of motion is discretized to a two-degree-of-freedom system. With a nondimensional time, an appropriate transformation, and a suitable coordinate scaling, the discretized integro–ordinary differential equations are transformed to those in terms of more convenient generalized coordinates. With use of the method of variation of parameters, the transformed equations are converted to equations in amplitudes and phases. For small excitation intensity, system damping, and material relaxation measure, the amplitude and phase equations are then approximated to a system of Itô equations, whose solution is a diffusive Markov process by making use of the stochastic averaging method. Through a polar coordinate transformation and by making use of Itô's lemma and Khasminskii's[19] technique, expressions for the largest Lyapunov exponent are obtained analytically. Some numerical results are presented to give a quantitative picture of the effect of the excitation intensity on the stability boundaries for a band-limited excitation with a nonzero spectrum, only in a neighborhood of some frequency ω_0 with a narrow frequency bandwidth $\omega_0 - \Delta\omega_0/2 < \omega < \omega_0 + \Delta\omega_0/2$, where $\Delta\omega_0 \ll \omega_0$. The effect of the spectral density of a white noise excitation on the stability boundaries is also presented.

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